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An introduction to Leonard pairs and Leonard systems

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Abstract

Let $\mathcal{F}$ denote a field, and let V denote a finite dimensional vector space over $\mathcal{F}$. We consider an ordered pair (A, A^*), where A and A^* are $\mathcal{F}$-linear transformations from V to V that satisfy conditions (i), (ii) below:

(i) There exists a basis for V with respect to which the matrix representing A is diagonal, and the matrix representing A^* is irreducible tridiagonal.

(ii) There exists a basis for V with respect to which the matrix representing A^* is diagonal, and the matrix representing A is irreducible tridiagonal.

We call such a pair a *Leonard pair* on V. We present a classification of Leonard pairs. We obtain Leonard pairs from irreducible representations of the quantum Lie algebra $U_q(sl_2)$. We show any Leonard pair satisfy two polynomial relations called the Askey-Wilson relations. We obtain Leonard pairs from five families of classical posets.

1 Introduction

Throughout this talk, $\mathcal{F}$ will denote an arbitrary field.

Definition 1.1 *Let V denote a finite dimensional vector space over $\mathcal{F}$. By a Leonard pair on V, we mean an ordered pair (A, A^*), where A and A^* are $\mathcal{F}$-linear transformations from V to V satisfying (i), (ii) below.*

(i) There exists a basis for V with respect to which the matrix representing A^* is diagonal, and the matrix representing A is irreducible tridiagonal.

(ii) There exists a basis for V with respect to which the matrix representing A is diagonal, and the matrix representing A^* is irreducible tridiagonal.

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(A tridiagonal matrix is said to be irreducible whenever all entries immediately above and below the main diagonal are nonzero).

Here is an example of a Leonard pair. Set $V = \mathcal{F}^4$ (column vectors), set

$$A = \begin{pmatrix} 0 & 3 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 3 & 0 \end{pmatrix}, \qquad A^* = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix},$$

and view A and A^* as linear transformations on V. We assume the characteristic of $\mathcal{F}$ is not 2 or 3, to insure A is irreducible. Then (A, A^*) is a Leonard pair on V. Indeed, condition (i) of Definition 1.1 is satisfied by the basis for V consisting of the columns of the 4 by 4 identity matrix. To verify condition (ii), we display an invertible matrix P such that $P^{-1}AP$ is diagonal, and such that $P^{-1}A^*P$ is irreducible tridiagonal. Put

$$P = \begin{pmatrix} 1 & 3 & 3 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -3 & 3 & -1 \end{pmatrix}.$$

By matrix multiplication $P^2 = 8I$, so P^{-1} exists. Also by matrix multiplication,

$$AP = PA^*.$$

Apparently $P^{-1}AP$ equals A^*, and is therefor diagonal. By the above line, and since P^{-1} is a scalar multiple of P, we find $P^{-1}A^*P$ equals A, and is therefor irreducible tridiagonal. Now condition (ii) of Definition 1.1 is satisfied by the basis for V consisting of the columns of P.

Referring to the above example, apparently the eigenvalues of A^* (and A) are $3, 1, -1, -3$, and we observe these are distinct. This will always be the case. In fact, it is an easy exercise to show the following.

Lemma 1.2 *With reference to Definition 1.1, let (A, A^*) denote a Leonard pair on V. Then the eigenvalues of A are distinct, and contained in $\mathcal{F}$. Moreover, the eigenvalues of A^* are distinct, and contained in $\mathcal{F}$.*

When studying Leonard pairs, it is often convenient to consider a related and somewhat more abstract object, which we call a *Leonard system.* To define this, we need a few terms. Let d denote a nonnegative integer, and let $\mathrm{Mat}_{d+1}(\mathcal{F})$ denote the $\mathcal{F}$-algebra consisting of all $d+1$ by $d+1$ matrices with entries in $\mathcal{F}$. We view the rows and columns as indexed by $0, 1, \ldots, d$. For the rest of this talk, $\mathcal{A}$ will denote an $\mathcal{F}$-algebra isomorphic to $\mathrm{Mat}_{d+1}(\mathcal{F})$. An element $A \in \mathcal{A}$ will be called *multiplicity-free* whenever it has $d+1$ distinct eigenvalues, all of which are in $\mathcal{F}$. Assume A is multiplicity free, and let $\mathcal{D}$ denote the subalgebra of $\mathcal{A}$ generated by A. Then $\mathcal{D}$ has a basis $E_0, E_1, \ldots, E_d$ such that

$$E_iE_j = \delta_{ij}E_i \qquad (0 \le i, j \le d),$$
$$\sum_{i=0}^{d} E_i = I.$$

The elements $E_0, E_1, \ldots, E_d$ are unique up to permutation, and are called the *primitive idempotents* of A.

Definition 1.3 *Let d denote a nonnegative integer, let $\mathcal{F}$ denote a field, and let $\mathcal{A}$ denote an $\mathcal{F}$-algebra isomorphic to $Mat_{d+1}(\mathcal{F})$. By a Leonard System in $\mathcal{A}$, we mean a sequence*

$$\Phi = (A;\, E_0,\, E_1,\, \ldots,\, E_d;\, A^*;\, E_0^*,\, E_1^*,\, \ldots,\, E_d^*) \tag{1}$$

that satisfies (i)–(v) below.

(i) $A,\ A^*$ *are both multiplicity-free elements in* $\mathcal{A}$.

(ii) $E_0,\, E_1,\, \ldots,\, E_d$ *is an ordering of the primitive idempotents of* A.

(iii) $E_0^*,\, E_1^*,\, \ldots,\, E_d^*$ *is an ordering of the primitive idempotents of* A^*.

(iv) $E_i A^* E_j = \begin{cases} 0, & \text{if } |i-j| > 1; \\ \neq 0, & \text{if } |i-j| = 1 \end{cases} \qquad (0 \le i, j \le d).$

(v) $E_i^* A E_j^* = \begin{cases} 0, & \text{if } |i-j| > 1; \\ \neq 0, & \text{if } |i-j| = 1 \end{cases} \qquad (0 \le i, j \le d).$

We refer to d as the diameter of Φ, and say Φ is over $\mathcal{F}$.

To see the connection between Leonard pairs and Leonard systems, observe conditions (ii), (iv) above assert that with respect to an appropriate basis consisting of eigenvectors for A, the matrix representing A^* is irreducible tridiagonal. Similarily, conditions (iii), (v) assert that with respect to an appropriate basis consisting of eigenvectors for A^*, the matrix representing A is irreducible tridiagonal.

Definition 1.4 *Let the Leonard system Φ be as in (1). We let θ_i (resp. θ_i^*) denote the eigenvalue of A (resp. A^*) associated with E_i (resp. E_i^*), for $0 \le i \le d$. We call $\theta_0, \theta_1, \ldots, \theta_d$ the eigenvalue sequence of Φ. We call $\theta_0^*, \theta_1^*, \ldots, \theta_d^*$ the dual eigenvalue sequence of Φ.*

Given a Leonard system

$$\Phi = (A; E_0, E_1, \ldots, E_d; A^*; E_0^*, E_1^*, \ldots, E_d^*),$$

we can get more Leonard systems. For example

$$\begin{aligned}
\Phi^* &:= (A^*;\, E_0^*,\, E_1^*,\, \ldots,\, E_d^*;\, A;\, E_0,\, E_1,\, \ldots,\, E_d),\\
\Phi^{\downarrow} &:= (A;\, E_0,\, E_1,\, \ldots,\, E_d;\, A^*;\, E_d^*,\, E_{d-1}^*,\, \ldots,\, E_0^*),\\
\Phi^{\Downarrow} &:= (A;\, E_d,\, E_{d-1},\, \ldots,\, E_0;\, A^*;\, E_0^*,\, E_1^*,\, \ldots,\, E_d^*)
\end{aligned}$$

are Leonard systems. Viewing $*, \downarrow, \Downarrow$ as permutations on the set of all Leonard systems,

$$\begin{gathered}
*^2 = \downarrow^2 = \Downarrow^2 = 1,\\
\downarrow * = * \Downarrow, \qquad \downarrow\Downarrow = \Downarrow\downarrow .
\end{gathered}$$

The group generated by symbols $*$, $\downarrow$, $\Downarrow$ subject to the above relations is the dihedral group D_4. We recall D_4 is the group of symmetries of a square, and has 8 elements. Apparently $*, \downarrow, \Downarrow$ induce an action of D_4 on the set of all Leonard systems. We say two Leonard systems are *relatives* whenever they are in the same orbit of this D_4 action.

In view of our above comments, when we discuss Leonard systems, we are often not interested in the orderings of the primitive idempotents involved; we just care how A and A^* interact. This brings us back to the notion of a Leonard pair.

Definition 1.5 *Let d denote a nonnegative integer, let $\mathcal{F}$ denote a field, and let $\mathcal{A}$ denote an $\mathcal{F}$-algebra isomorphic to $Mat_{d+1}(\mathcal{F})$. By a Leonard pair in $\mathcal{A}$, we mean an ordered pair (A, A^*) such that*

(i) A, A^* are both multiplicity free elements of $\mathcal{A}$, and

(ii) There exists an ordering E_0, E_1, ..., E_d of the primitive idempotents of A, and there exists an ordering E^*_0, E^*_1, ..., E^*_d of the primitive idempotents of A^*, such that $(A;\ E_0,\ E_1,\ \ldots,\ E_d;\ A^*;\ E^*_0,\ E^*_1,\ \ldots,\ E^*_d)$ is a Leonard System.

2 A classification of Leonard systems

When studying a Leonard system Φ, it is often useful to examine a second Leonard system that is isomorphic to Φ but in a particularily nice form. We present such a 'canonical form'. To describe it, we use the following notation. Let

$$\Phi = (A; E_0, E_1, \ldots, E_d; A^*; E^*_0, E^*_1, \ldots, E^*_d)$$

denote a Leonard system in $\mathcal{A}$, and let $\sigma : \mathcal{A} \to \mathcal{A}'$ denote an isomorphism of $\mathcal{F}$-algebras. Then we write

$$\Phi^\sigma := (A^\sigma; E^\sigma_0, E^\sigma_1, \ldots, E^\sigma_d; A^{*\sigma}; E^{*\sigma}_0, E^{*\sigma}_1, \ldots, E^{*\sigma}_d),$$

and observe Φ^σ is a Leonard system in $\mathcal{A}'$.

Let us say a matrix $X \in \mathrm{Mat}_{d+1}(\mathcal{F})$ is *lower di-diagonal* whenever

$$X_{ij} \neq 0 \quad \to \quad i - j \in \{0, 1\} \qquad (0 \leq i, j \leq d).$$

That is, X is lower di-diagonal whenever each nonzero entry lies either on or immediately below the main diagonal. We say X is *upper di-diagonal* whenever the transpose X^t is lower di-diagonal.

Let Φ denote the Leonard system in (1). We say Φ is in *split canonical form* whenever (i)–(iii) hold below.

(i) $\mathcal{A} = \mathrm{Mat}_{d+1}(\mathcal{F})$.

(ii) A is lower di-diagonal, with $A_{i,i-1} = 1$ for $1 \leq i \leq d$, and $A_{ii} = \theta_i$ for $0 \leq i \leq d$, where θ_i denotes the eigenvalue of A associated with E_i.

(iii) A^* is upper di-diagonal, with $A^*_{ii} = \theta^*_i$ for $0 \leq i \leq d$, where θ^*_i denotes the eigenvalue of A^* associated with E^*_i.

We show there there exists a unique isomorphism of $\mathcal{F}$-algebras $\heartsuit : \mathcal{A} \rightarrow \mathrm{Mat}_{d+1}(\mathcal{F})$ such that $\Phi^\heartsuit$ is in split canonical form. Apparently

$$A^\heartsuit = \begin{pmatrix} \theta_0 & & & & & 0 \\ 1 & \theta_1 & & & & \\ & 1 & \theta_2 & & & \\ & & \cdot & \cdot & & \\ & & & \cdot & \cdot & \\ 0 & & & & 1 & \theta_d \end{pmatrix}, \qquad A^{*\heartsuit} = \begin{pmatrix} \theta^*_0 & \varphi_1 & & & & 0 \\ & \theta^*_1 & \varphi_2 & & & \\ & & \theta^*_2 & \cdot & & \\ & & & \cdot & \cdot & \\ & & & & \cdot & \varphi_d \\ 0 & & & & & \theta^*_d \end{pmatrix},$$

where $\varphi_1, \varphi_2, \ldots, \varphi_d$ are appropriate scalars in $\mathcal{F}$. We call $\varphi_1, \varphi_2, \ldots, \varphi_d$ the *φ-sequence* of Φ. Let $\phi_1, \phi_2, \ldots, \phi_d$ denote the φ-sequence for $\Phi^\Downarrow$. Then abbreviating $\diamondsuit := \heartsuit(\Phi^\Downarrow)$, we have

$$A^\diamondsuit = \begin{pmatrix} \theta_d & & & & & 0 \\ 1 & \theta_{d-1} & & & & \\ & 1 & \theta_{d-2} & & & \\ & & \cdot & \cdot & & \\ & & & \cdot & \cdot & \\ 0 & & & & 1 & \theta_0 \end{pmatrix}, \qquad A^{*\diamondsuit} = \begin{pmatrix} \theta^*_0 & \phi_1 & & & & 0 \\ & \theta^*_1 & \phi_2 & & & \\ & & \theta^*_2 & \cdot & & \\ & & & \cdot & \cdot & \\ & & & & \cdot & \phi_d \\ 0 & & & & & \theta^*_d \end{pmatrix}.$$

We call $\phi_1, \phi_2, \ldots, \phi_d$ the *ϕ-sequence* of Φ.

We obtain the following classification of Leonard systems.

Theorem 2.1 *[7] Let d denote a nonnegative integer, let $\mathcal{F}$ denote a field, and let*

$$\theta_0, \theta_1, \ldots, \theta_d; \qquad \theta^*_0, \theta^*_1, \ldots, \theta^*_d;$$
$$\varphi_1, \varphi_2, \ldots, \varphi_d; \qquad \phi_1, \phi_2, \ldots, \phi_d$$

denote scalars in $\mathcal{F}$. Then there exists a Leonard System Φ over $\mathcal{F}$ with eigenvalue sequence $\theta_0, \theta_1, \ldots, \theta_d$, dual eigenvalue sequence $\theta^*_0, \theta^*_1, \ldots, \theta^*_d$, φ-sequence $\varphi_1, \varphi_2, \ldots, \varphi_d$, and ϕ-sequence $\phi_1, \phi_2, \ldots, \phi_d$ if and only if (i)–(v) hold below.

(i) $\varphi_i \neq 0, \qquad \phi_i \neq 0 \qquad\qquad (1 \leq i \leq d),$

(ii) $\theta_i \neq \theta_j, \qquad \theta^*_i \neq \theta^*_j \qquad \text{if } i \neq j, \qquad\qquad (0 \leq i, j \leq d),$

(iii) $\displaystyle \varphi_i = \phi_1 \sum_{h=0}^{i-1} \frac{\theta_h - \theta_{d-h}}{\theta_0 - \theta_d} + (\theta^*_i - \theta^*_0)(\theta_{i-1} - \theta_d) \qquad (1 \leq i \leq d),$

(iv) $\displaystyle \phi_i = \varphi_1 \sum_{h=0}^{i-1} \frac{\theta_h - \theta_{d-h}}{\theta_0 - \theta_d} + (\theta^*_i - \theta^*_0)(\theta_{d-i+1} - \theta_0) \qquad (1 \leq i \leq d),$

(v) The expressions

$$\frac{\theta_{i-2}-\theta_{i+1}}{\theta_{i-1}-\theta_i}, \qquad \frac{\theta^*_{i-2}-\theta^*_{i+1}}{\theta^*_{i-1}-\theta^*_i} \tag{2}$$

are equal and independent of i, *for* $2 \le i \le d-1$.

Moreover, if (i)–(v) hold above then Φ *is unique up to isomorphism of Leonard Systems.*

From the above theorem, we routinely obtain the following corollary.

Corollary 2.2 *[7] Let* d *denote a nonnegative integer, and let* $\mathcal{F}$ *denote a field. Let* A *and* A^* *denote any matrices in* $Mat_{d+1}(\mathcal{F})$ *of the form*

$$A = \begin{pmatrix} \theta_0 & & & & & 0 \\ 1 & \theta_1 & & & & \\ & 1 & \theta_2 & & & \\ & & \cdot & \cdot & & \\ & & & \cdot & \cdot & \\ 0 & & & & 1 & \theta_d \end{pmatrix}, \qquad A^* = \begin{pmatrix} \theta^*_0 & \varphi_1 & & & & 0 \\ & \theta^*_1 & \varphi_2 & & & \\ & & \theta^*_2 & \cdot & & \\ & & & \cdot & \cdot & \\ & & & & \cdot & \varphi_d \\ 0 & & & & & \theta^*_d \end{pmatrix}.$$

Then the following are equivalent.

(i) (A, A^*) *is a Leonard pair.*

(ii) There exist scalars $\phi_1, \phi_2, \ldots, \phi_d$ *in* $\mathcal{F}$ *such that conditions (i)–(v) hold in Theorem 2.1.*

3 The quantum Lie algebra $U_q(sl_2)$

In this section, we obtain Leonard pairs from irreducible representations of the quantum Lie algebra $U_q(sl_2)$. Throughout this section, we assume our ground field $\mathcal{F}$ is algebraically closed with characteristic zero. We let q denote a nonzero element in $\mathcal{F}$, and assume q is not a root of 1.

Recall $U_q(sl_2)$ is the associative $\mathcal{F}$-algebra with 1 generated by symbols e, f, k, k^{-1} subject to the relations

$$kk^{-1} = k^{-1}k = 1, \tag{3}$$

$$ke = q^2 ek, \qquad kf = q^{-2} fk, \tag{4}$$

$$ef - fe = \frac{k-k^{-1}}{q-q^{-1}}. \tag{5}$$

Let d denote a nonnegative integer, and put

$$E = \begin{pmatrix} 0 & [d] & & & \mathbf{0} \\ & 0 & [d-1] & & \\ & & 0 & \ddots & \\ & & & \ddots & [1] \\ \mathbf{0} & & & & 0 \end{pmatrix}, \qquad F = \begin{pmatrix} 0 & & & & \mathbf{0} \\ [1] & 0 & & & \\ & [2] & 0 & & \\ & & \ddots & \ddots & \\ \mathbf{0} & & & [d] & 0 \end{pmatrix},$$

where

$$[i] = \frac{q^i - q^{-i}}{q - q^{-1}} \qquad (\forall i \in \mathbb{Z}).$$

Also put

$$K = \operatorname{diag}(q^d, q^{d-2}, q^{d-4}, \ldots, q^{-d}).$$

Then K is invertible, and E, F, K satisfy the equations (4), (5), so they support a representation of $U_q(sl_2)$. It can be shown the representation is irreducible.

Let α and α^* denote nonzero elements in $\mathcal{F}$ such that $\alpha\alpha^*$ is not a power of q, and put

$$\begin{aligned} A &= \alpha F + \frac{K}{q - q^{-1}}, \\ A^* &= \alpha^* E + \frac{K^{-1}}{q - q^{-1}}. \end{aligned}$$

We claim (A, A^*) is a Leonard pair. To see this, let σ denote the automorphism of $\mathrm{Mat}_{d+1}(\mathcal{F})$ satisfying

$$X^\sigma = D^{-1} X D \qquad (\forall X \in \mathrm{Mat}_{d+1}(\mathcal{F})),$$

where D is the diagonal matrix in $\mathrm{Mat}_{d+1}(\mathcal{F})$ with entries

$$D_{ii} = [1][2]\cdots[i]\alpha^i \qquad (0 \le i \le d).$$

Then

$$A^\sigma = \begin{pmatrix} \theta_0 & & & & \mathbf{0} \\ 1 & \theta_1 & & & \\ & 1 & \theta_2 & & \\ & & \ddots & \ddots & \\ \mathbf{0} & & & 1 & \theta_d \end{pmatrix}, \qquad A^{*\sigma} = \begin{pmatrix} \theta_0^* & \varphi_1 & & & \mathbf{0} \\ & \theta_1^* & \varphi_2 & & \\ & & \theta_2^* & \ddots & \\ & & & \ddots & \varphi_d \\ \mathbf{0} & & & & \theta_d^* \end{pmatrix},$$

where

$$\theta_i = \frac{q^{d-2i}}{q - q^{-1}}, \qquad (0 \le i \le d), \tag{6}$$

$$\theta_i^* = \frac{q^{2i-d}}{q - q^{-1}}, \qquad (0 \le i \le d), \tag{7}$$

$$\varphi_i = [i][d-i+1]\alpha\alpha^* \qquad (1 \le i \le d). \tag{8}$$

Set

$$\phi_i = \varphi_1 \sum_{h=0}^{i-1} \frac{\theta_h - \theta_{d-h}}{\theta_0 - \theta_d} + (\theta^*_i - \theta^*_0)(\theta_{d-i+1} - \theta_0) \qquad (1 \le i \le d).$$

Evaluating this using (6)–(8), we obtain

$$\phi_i = [i][d-i+1](\alpha\alpha^* - q^{2i-d-1}) \qquad (1 \le i \le d).$$

One readily checks the above scalars $\theta_i, \theta^*_i, \varphi_i, \phi_i$ satisfy the conditions of Theorem 2.1, so $(A^\sigma, A^{*\sigma})$ is a Leonard pair by Corollary 2.2. Applying σ^{-1}, we find (A, A^*) is a Leonard pair. For this example, it turns out

$$\begin{aligned} A^2A^* - (q^2+q^{-2})AA^*A + A^*A^2 &= \omega A + \eta I, \\ A^{*2}A - (q^2+q^{-2})A^*AA^* + AA^{*2} &= \omega A^* + \eta^* I, \end{aligned}$$

where

$$\eta = \alpha\alpha^* \frac{q+q^{-1}}{q-q^{-1}}, \qquad \eta^* = \alpha\alpha^* \frac{q+q^{-1}}{q-q^{-1}},$$
$$\omega = -1 - \alpha\alpha^*(q^{-d-1} + q^{d+1}). \tag{9}$$

We comment there is a second Leonard pair associated with $U_q(sl_2)$. Let α, α^* be as above, and put

$$\begin{aligned} B &= \alpha K - (1-q^{-2})KE, \\ B^* &= \alpha^* K^{-1} - (1-q^{-2})K^{-1}F. \end{aligned}$$

Then (B, B^*) is a Leonard pair. The proof is similar, and omitted.

4 The Askey-Wilson relations

In the previous section, we obtained a Leonard pair whose elements A, A^* satisfied two polynomial equations. It turns out every Leonard pair satisfies a similar pair of equations.

Theorem 4.1 *[6] Let d denote a nonnegative integer, let $\mathcal{F}$ denote any field, and let $\mathcal{A}$ denote an $\mathcal{F}$-algebra isomorphic to $Mat_{d+1}(\mathcal{F})$. Let (A, A^*) denote a Leonard pair in $\mathcal{A}$. Then there exists a sequence of scalars $\beta, \gamma, \gamma^*, \varrho, \varrho^*, \omega, \eta, \eta^*$ from $\mathcal{F}$ such that*

$$\begin{aligned} A^2A^* - \beta AA^*A + A^*A^2 - \gamma(AA^* + A^*A) - \varrho A^* &= \gamma^* A^2 + \omega A + \eta I, \\ A^{*2}A - \beta A^*AA^* + AA^{*2} - \gamma^*(A^*A + AA^*) - \varrho^* A &= \gamma A^{*2} + \omega A^* + \eta^* I. \end{aligned}$$

The sequence is unique if $d \ge 3$.

The above equations are known as the *Askey-Wilson* relations [1], [2], [3], [4], [5], [9], [10], [11].

Concerning the converse to the above theorem, we have the following.

Theorem 4.2 *[6] Let d denote a nonnegative integer, let $\mathcal{F}$ denote any field, and let $\mathcal{A}$ denote an $\mathcal{F}$-algebra isomorphic to $Mat_{d+1}(\mathcal{F})$. Let A, A^* denote multiplicity free elements in $\mathcal{A}$, and assume the irreducible $\mathcal{A}$-module is irreducible as an (A, A^*)-module. Pick any scalars $\beta, \gamma, \gamma^*, \varrho, \varrho^*, \omega, \eta, \eta^*$ from $\mathcal{F}$, and assume A, A^* satisfy the corresponding Askey-Wilson relations. Assume further that none of the following (i)–(iii) occur:*

(i) q *is a primitive* $d+1^{st}$ *root of* 1, *where* $q+q^{-1}=\beta$.

(ii) $\beta = 2$ *and* $d+1 = char(\mathcal{F})$.

(iii) $\beta = -2$ *and* $d+1 = 2\,char(\mathcal{F})$.

Then (A, A^*) is a Leonard pair in $\mathcal{A}$.

5 Leonard pairs from the classical posets

There is a way to obtain Leonard pairs from the following classical posets: (i) the subset lattice, (ii) the subspace lattice, (iii) the Hamming semi-lattice, (iv) the attenuated spaces, (v) the classical polar spaces. For the definitions of these posets, see [8]. The argument in each case is similar. To illustrate it, we will consider the attenuated spaces in some detail.

Definition 5.1 *Let $\mathcal{F}$ denote any field, let V denote a finite dimensional vector space over $\mathcal{F}$, and let A and A^* denote $\mathcal{F}$-linear transformations from V to V. We say (A, A^*) is a generalized Leonard pair on V whenever there exists a decomposition*

$$V = V_1 + V_2 + \cdots + V_n \qquad (\textit{direct sum}),$$

such that

$$AV_i \subseteq V_i, \qquad A^*V_i \subseteq V_i, \qquad (1 \leq i \leq n)$$

and such that

$$(A|_{V_i}, A^*|_{V_i}) \qquad \textit{is a Leonard Pair} \qquad (1 \leq i \leq n).$$

The posets mentioned above all support generalized Leonard pairs. In each case, the underlying vector space V has the following form. Let X denote a finite set. By $\mathcal{F}X$, we mean the vector space over $\mathcal{F}$ consisting of all formal sums

$$\sum_{x \in X} \alpha_x x,$$

where $\alpha_x \in \mathcal{F}$ for all $x \in X$.

We will be discussing posets, so let us recall some terms. let P denote a poset. For all $x, y \in P$, we say y *covers* x whenever $x < y$, and there does not exist $z \in P$ such that $x < z < y$. In this case, we write $x \prec y$. Let L denote the matrix in $\mathrm{Mat}_P(\mathbb{C})$ with entries

$$L_{xy} = \begin{cases} 1, & \text{if } x \prec y; \\ 0, & \text{if } x \not\prec y \end{cases} \qquad (\forall x, y \in P).$$

Viewing L as a linear transformation on $\mathbb{C}P$,

$$Lx = \sum_{\substack{y \in P \\ y \prec x}} y \qquad (\forall x \in P).$$

We call L the *lowering matrix* on P. Let R denote the matrix in $\mathrm{Mat}_P(\mathbb{C})$ with entries

$$R_{xy} = \begin{cases} 1, & \text{if } y \prec x; \\ 0, & \text{if } y \not\prec x \end{cases} \qquad (\forall x, y \in P).$$

Viewing R as a linear transformation on $\mathbb{C}P$,

$$Rx = \sum_{\substack{y \in P \\ x \prec y}} y \qquad (\forall x \in P).$$

We call R the *raising matrix* on P. Now assume P is ranked, with rank denoted N. For $0 \le i \le N$, let F_i denote the diagonal matrix in $\mathrm{Mat}_P(\mathbb{C})$ with yy entry

$$(F_i)_{yy} = \begin{cases} 1, & \text{if } \mathrm{rank}(y) = i; \\ 0, & \text{if } \mathrm{rank}(y) \ne i \end{cases} \qquad (\forall y \in P).$$

We refer to F_i as the i^{th} *projection matrix* of P. We observe

$$F_i F_j = \delta_{ij} F_i \qquad (0 \le i, j \le N),$$
$$F_0 + F_1 + \cdots + F_N = I.$$

Moreover,

$$F_i V = \mathrm{Span}\{x \in P \mid \mathrm{rank}(x) = i\} \qquad (0 \le i \le N),$$

where $V = \mathbb{C}P$.

For each of the five families of classical posets we mentioned at the outset, we obtain generalized Leonard pairs on $V = \mathbb{C}P$ of the form

$$A = \alpha R + \sum_{i=0}^{N} \theta_i F_i, \qquad (10)$$

$$A^* = \alpha^* L + \sum_{i=0}^{N} \theta_i^* F_i, \qquad (11)$$

where the $\alpha, \alpha^*, \theta_i, \theta_i^*$ are complex scalars.

To illustrate, we now restrict our attention to the attenuated space poset $A_q(N, M)$. This poset is defined as follows. Let M and N denote nonnegative integers, let H denote a vector space of dimension $M+N$ over $GF(q)$, and fix a subspace $h \subseteq H$ of dimension M. Let P denote the poset consisting of all subspaces x of H such that $x \cap h = 0$. The partial order on P is

$$x \leq y \quad \text{whenever} \quad x \subseteq y \qquad (\forall x, y \in P).$$

The poset P is ranked, with

$$\operatorname{rank}(x) = \dim(x) \qquad (\forall x \in P).$$

Apparently, P has rank N. For $0 \leq i \leq N$, each rank i element of P covers exactly

$$\frac{q^i - 1}{q - 1}$$

elements of P, and is covered by exactly

$$\frac{q^{N+M-i} - q^M}{q - 1}$$

elements in P. Moreover, it is shown in [8] that

$$\frac{q}{q+1} RL^2 - LRL + \frac{1}{q+1} L^2 R + f_i L \tag{12}$$

vanishes on $F_i V$, where R and L are the raising and lowering matrices, where $V = \mathbb{C}P$, and where

$$f_i = q^{N+M-i}. \tag{13}$$

Put

$$A = R + \sum_{i=0}^{N} \frac{q^i}{q-1} F_i, \tag{14}$$

$$A^* = \alpha^* L + \sum_{i=0}^{N} \frac{q^{-i}}{q-1} F_i, \tag{15}$$

where α^* is any scalar in $\mathbb{C}$ that is not one of $q^{-M-1}, q^{-M-2}, \ldots, q^{-M-N}$. We show (A, A^*) is a generalized Leonard pair on V. Let T denote the subalgebra of $\mathrm{Mat}_P(\mathbb{C})$ generated by $R, L, F_0, F_1, \ldots, F_N$. Observe $R^t = L$, and each of $F_0, F_1, \ldots, F_N$ is symmetric, so T is closed under the conjugate-transpose map. It follows T is semi-simple, so V is a direct sum of irreducible T-submodules. Let W denote an irreducible T-submodule of V. The matrices A and A^* are contained in T by (14), (15), so

$$AW \subseteq W, \qquad A^*W \subseteq W.$$

It remains to show that

$$(A|_W, A^*|_W)$$

is a Leonard pair on W. We do this as follows. Using (12), one can show there exists integers r,p $(0 \leq r \leq p \leq N)$ and a basis $w_r, w_{r+1}, \ldots, w_p$ for W such that

(i) $w_i \in F_i V \qquad (r \leq i \leq p)$,

(ii) $Rw_i = w_{i+1} \qquad (r \leq i < p), \quad Rw_p = 0$,

(iii) $Lw_i = x_i(r,p) w_{i-1} \qquad (r < i \leq p), \quad Lw_r = 0$,

where

$$x_i(r,p) = \frac{q^{M+N-r-p-i+1}(q^i - q^r)(q^p - q^{i-1})}{(q-1)^2} \tag{16}$$

for $r < i \leq p$. Let B (resp. B^*) denote the matrix representing A (resp. A^*) with respect to the basis $w_r, w_{r+1}, \ldots, w_p$. Apparently

$$B = \begin{pmatrix} \theta_0 & & & & & 0 \\ 1 & \theta_1 & & & & \\ & 1 & \theta_2 & & & \\ & & \cdot & \cdot & & \\ & & & \cdot & \cdot & \\ 0 & & & & 1 & \theta_d \end{pmatrix}, \qquad B^* = \begin{pmatrix} \theta_0^* & \varphi_1 & & & & 0 \\ & \theta_1^* & \varphi_2 & & & \\ & & \theta_2^* & \cdot & & \\ & & & \cdot & \cdot & \\ & & & & \cdot & \varphi_d \\ 0 & & & & & \theta_d^* \end{pmatrix},$$

where $d = p - r$, and where

$$\theta_i = \frac{q^{r+i}}{q-1}, \qquad \theta_i^* = \frac{q^{-r-i}}{q-1} \qquad (0 \leq i \leq d), \tag{17}$$

$$\varphi_i = \alpha^* x_{r+i}(r,p) \qquad (1 \leq i \leq d). \tag{18}$$

Set

$$\phi_i = \varphi_1 \sum_{h=0}^{i-1} \frac{\theta_h - \theta_{d-h}}{\theta_0 - \theta_d} + (\theta_i^* - \theta_0^*)(\theta_{d-i+1} - \theta_0) \qquad (1 \leq i \leq d).$$

Evaluating this using (16), (17), and (18) we obtain

$$\phi_i = -\frac{(1-q^i)(1-q^{d-i+1})(1-\alpha^* q^{M+N+i-r-d})}{(q-1)^2 q^i} \qquad (1 \leq i \leq d).$$

One readily checks the above scalars $\theta_i, \theta_i^*, \varphi_i, \phi_i$ satisfy the conditions of Theorem 2.1, so (B, B^*) is a Leonard pair in $\mathrm{Mat}_{d+1}(\mathcal{F})$ by Corollary 2.2. It follows

$$(A|_W, A^*|_W)$$

is a Leonard pair on W. We have now shown (A, A^*) is a generalized Leonard pair on V. We remark that by (12), (14), (15), we have

$$[A, A^2A^* - (q+q^{-1})AA^*A + A^*A^2] = 0,$$
$$[A^*, A^{*2}A - (q+q^{-1})A^*AA^* + AA^{*2}] = 0$$

for this example.

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